Displacement modes on a string

The 1-D wave eq. provides general solutions of the form

$$y(t,x) = g\left(t-\frac{x}{v}\right) + h\left(t+\frac{x}{v}\right)$$

where $g()$ and $h()$ are arbitrary waveforms and v is the wave speed on the string. If we consider one frequency, $\omega = 2\pi f$, we have a general sinusoidal solution:

$$y(t,x) = y^{(+)} \cos\left(\omega\left(t-\frac{x}{v}\right)+\phi^{(+)}\right) + y^{(-)}\cos\left(\omega\left(t+\frac{x}{v}\right)+\phi^{(-)}\right)$$

$$= \mathrm{Re}\left\{ \underline{y}^{(+)} e^{j(\omega(t-\frac{x}{v}))} + \underline{y}^{(-)} e^{j(\omega(t+\frac{x}{v}))}\right\}$$

where $\underline{y}^{(+)} = y^{(+)}e^{j\phi^{(+)}}$ and $\underline{y}^{(-)} = y^{(-)}e^{j\phi^{(-)}}$ are the complex amplitudes of the positive and negative travelling waves, and $y(t,x) = \mathrm{Re}\{\underline{y}(t,x)\}$ is the real part of the complex displacement field on the string.

At resonance, wave interference creates a standing wave

$$\underline{y}(t,x) = X(x)\, e^{j\omega t}$$

To get the mode shape, $X(x)$, we must use the string's boundary conditions, $y_0 = y_L = 0$, which specify how the waves are reflected at the ends.

Mode shapes on a string

In complex form, we have the general sinusoidal behaviour:

$$y(t,x) = y^{(+)} e^{j(\omega(t-\frac{x}{v}))} + y^{(-)} e^{j(\omega(t+\frac{x}{v}))}$$

The first boundary condition at $x=0$ gives

$$y^{(+)} e^{j\omega t} + y^{(-)} e^{j\omega t} = 0$$

$$\Rightarrow \quad y^{(+)} = -y^{(-)}$$

This result simplifies the sinusoidal soundfield, via trigonometry:

$$y(t,x) = y^{(+)}\left(e^{-j\frac{\omega x}{v}} - e^{+j\frac{\omega x}{v}}\right) e^{j\omega t}$$

$$= -2jy^{(+)} \cdot \sin\left(\frac{\omega x}{v}\right) \cdot e^{j\omega t}$$

$$\boxed{\sin x = \frac{e^{jx} - e^{-jx}}{2j}}$$

The second boundary condition at $x=L$ gives

$$y^{(+)} e^{j\omega(t-\frac{L}{v})} - y^{(+)} e^{j\omega(t+\frac{L}{v})} = 0$$

$$\Rightarrow \quad y^{(+)}\left(e^{-j\frac{\omega L}{v}} - e^{+j\frac{\omega L}{v}}\right) e^{j\omega t} = 0$$

$$\Rightarrow \quad \left(e^{+j\frac{\omega L}{v}} - e^{-j\frac{\omega L}{v}}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\omega L}{v}\right) = 0$$

$$\Rightarrow \quad \frac{\omega L}{v} = \{0, \pi, 2\pi, \ldots\} = n\pi \quad \text{for } n = \{0, 1, 2, \ldots\}$$

Thus, for resonances with these boundary conditions, we obtain

$$\omega = \frac{n\pi v}{L} \quad \text{and} \quad f = \frac{nv}{2L}$$

Finally, $y(t,x) = -2jy^{(+)} \cdot \overbrace{\sin\left(\frac{n\pi x}{L}\right)}^{X(x)} \cdot e^{j\omega t}$