EE1.el3 (EEE1023): Electronics III

Acoustics lecture 17

Resonators and waveguides

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Acoustic devices

Objectives:

- derive mode shapes of standing waves
- identify the modes of propagating waves
- relate resonances to properties of acoustical artifacts

Topics:

- resonators
- waveguides

Preparation for acoustic devices

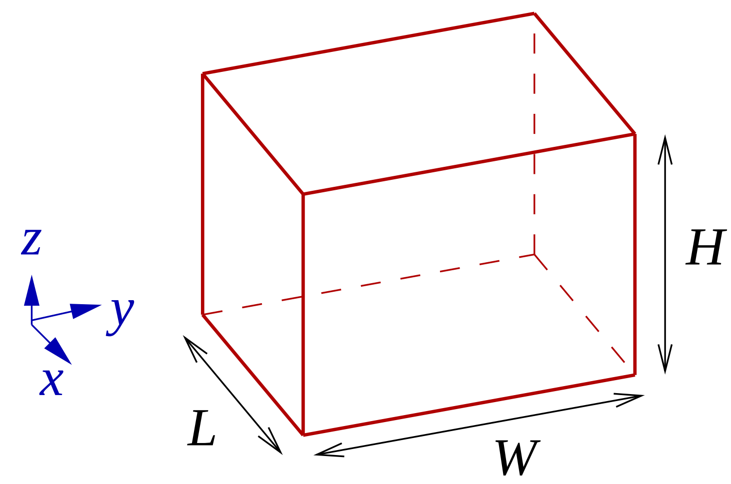
- What is an acoustic **resonator**?
  - find a definition
  - draw a sketch of an example

- What is an acoustic **waveguide**?
  - look up a definition
  - give an example

Cavities and waveguides

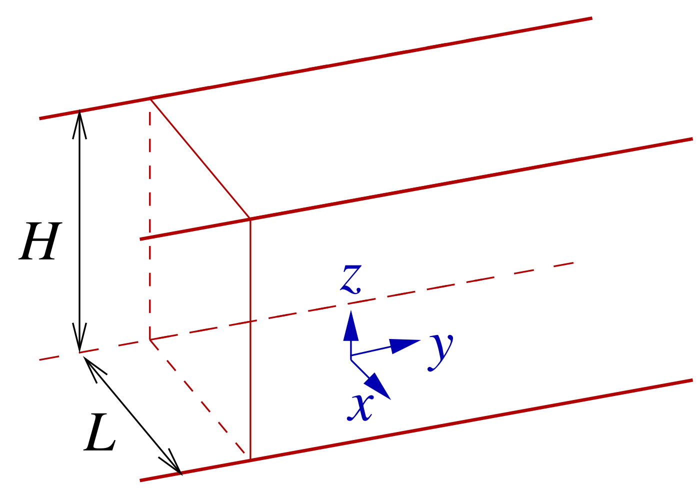
Cavity: accrual of acoustic energy in an enclosed space

e.g., a room, a silencer, or a loudspeaker cabinet



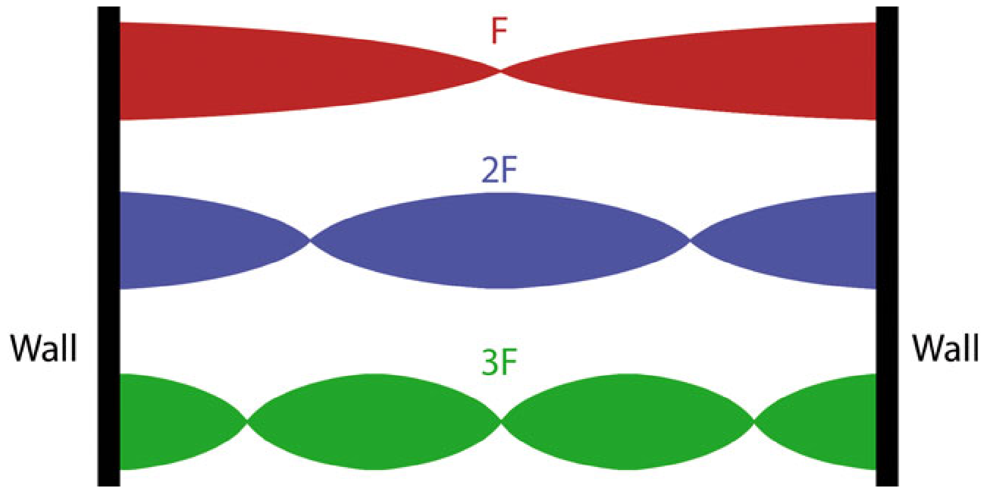
Waveguide: transmission of acoustic energy along a bounded channel

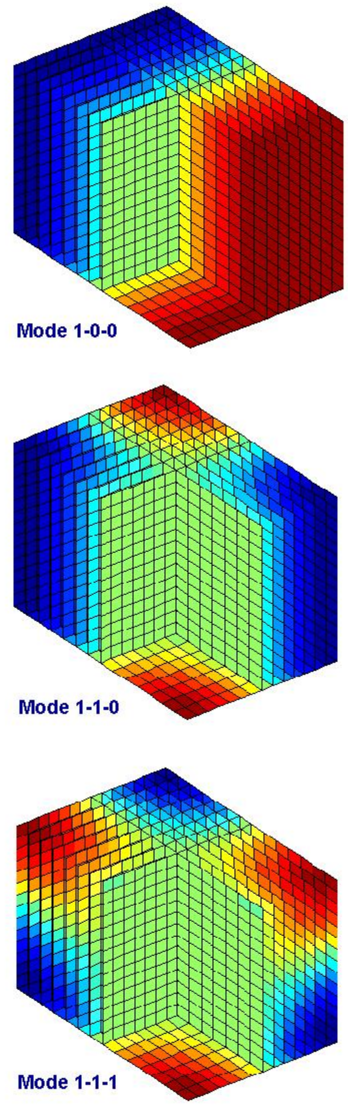
e.g., a system of ducting, a bass reflex port, sound in the atmosphere or oceans



Resonators

Modes in a cavity





Resonance frequencies of a cavity are defined by their mode number (l,m,n):

$$f(l,m,n) = \frac{c}{2}\sqrt{\left(\frac{l}{L}\right)^2 + \left(\frac{m}{W}\right)^2 + \left(\frac{n}{H}\right)^2}$$

Pressure modes in a closed pipe

The 1-D wave eq. provides general solutions of the form

$$p(t,x) = g\left(t - \frac{x}{c}\right) + h\left(t + \frac{x}{c}\right) \qquad (1)$$

At one frequency, $\omega = 2\pi f$, we have

$$\begin{aligned} p(t,x) &= p^{(+)} \cos\left(\omega\left(t - \frac{x}{c}\right) + \phi^{(+)}\right) + p^{(-)} \cos\left(\omega\left(t + \frac{x}{c}\right) + \phi^{(-)}\right) \\ &= \text{Re}\left\{\mathbf{p}^{(+)}\mathbf{e}^{j(\omega t - kx)} + \mathbf{p}^{(-)}\mathbf{e}^{j(\omega t + kx)}\right\} \qquad (2) \end{aligned}$$

where $\mathbf{p}^{(+)} = p^{(+)}\mathbf{e}^{j\phi^{(+)}}$ and $\mathbf{p}^{(-)} = p^{(-)}\mathbf{e}^{j\phi^{(-)}}$ are the complex amplitudes of the positive and negative travelling waves, $k = \omega/c$ is the wavenumber, and $p(t,x) = \text{Re}\{\mathbf{p}(t,x)\}$ is the real part of the complex pressure field in the pipe.

At resonance, wave interference creates a standing wave

$$\mathbf{p}(t,x) = X(x)\mathbf{e}^{j\omega t} \qquad (3)$$

To get the mode shape, $X(x)$, we must use the pipe's boundary conditions, which specify how waves are reflected at the ends.

Mode shapes in a closed pipe

In complex form, we have general sinusoidal soundfields

$$\mathbf{p}(t,x) = \mathbf{p}^{(+)}\mathbf{e}^{j(\omega t - kx)} + \mathbf{p}^{(-)}\mathbf{e}^{j(\omega t + kx)} \tag{4}$$

The boundary conditions specify the pressure gradient

$$\begin{aligned} \frac{\partial \mathbf{p}(t,x)}{\partial x} &= -jk\left(\mathbf{p}^{(+)}\mathbf{e}^{j(\omega t - kx)} - \mathbf{p}^{(-)}\mathbf{e}^{j(\omega t + kx)}\right) \\ &= -jk\left(\mathbf{p}^{(+)}\mathbf{e}^{-jkx} - \mathbf{p}^{(-)}\mathbf{e}^{+jkx}\right)\mathbf{e}^{j\omega t} \end{aligned} \tag{5}$$

At $x = 0$, the first boundary condition:

$$\begin{aligned} -jk\left(\mathbf{p}^{(+)} - \mathbf{p}^{(-)}\right)\mathbf{e}^{j\omega t} &= 0 \\ \mathbf{p}^{(+)} &= \mathbf{p}^{(-)} \end{aligned} \tag{6}$$

This result gives us soundfields that are even about $x = 0$,

$$\begin{aligned} \mathbf{p}(t,x) &= \mathbf{p}^{(+)}\left(\mathbf{e}^{-jkx} + \mathbf{e}^{+jkx}\right)\mathbf{e}^{j\omega t} \\ &= 2\mathbf{p}^{(+)}\cos(kx)\,\mathbf{e}^{j\omega t} \end{aligned} \tag{7}$$

Mode shapes in a closed pipe

At $x = L$, the second boundary condition:

$$\begin{aligned} -jk\left(\mathbf{p}^{(+)}\mathbf{e}^{-jkL} - \mathbf{p}^{(+)}\mathbf{e}^{+jkL}\right)\mathbf{e}^{j\omega t} &= 0 \\ \left(\mathbf{e}^{-jkL} - \mathbf{e}^{+jkL}\right) &= 0 \\ \sin(kL) &= 0 \end{aligned} \tag{8}$$

$$kL = n\pi \qquad \text{for } n = \{0, 1, 2, \ldots\} \tag{9}$$

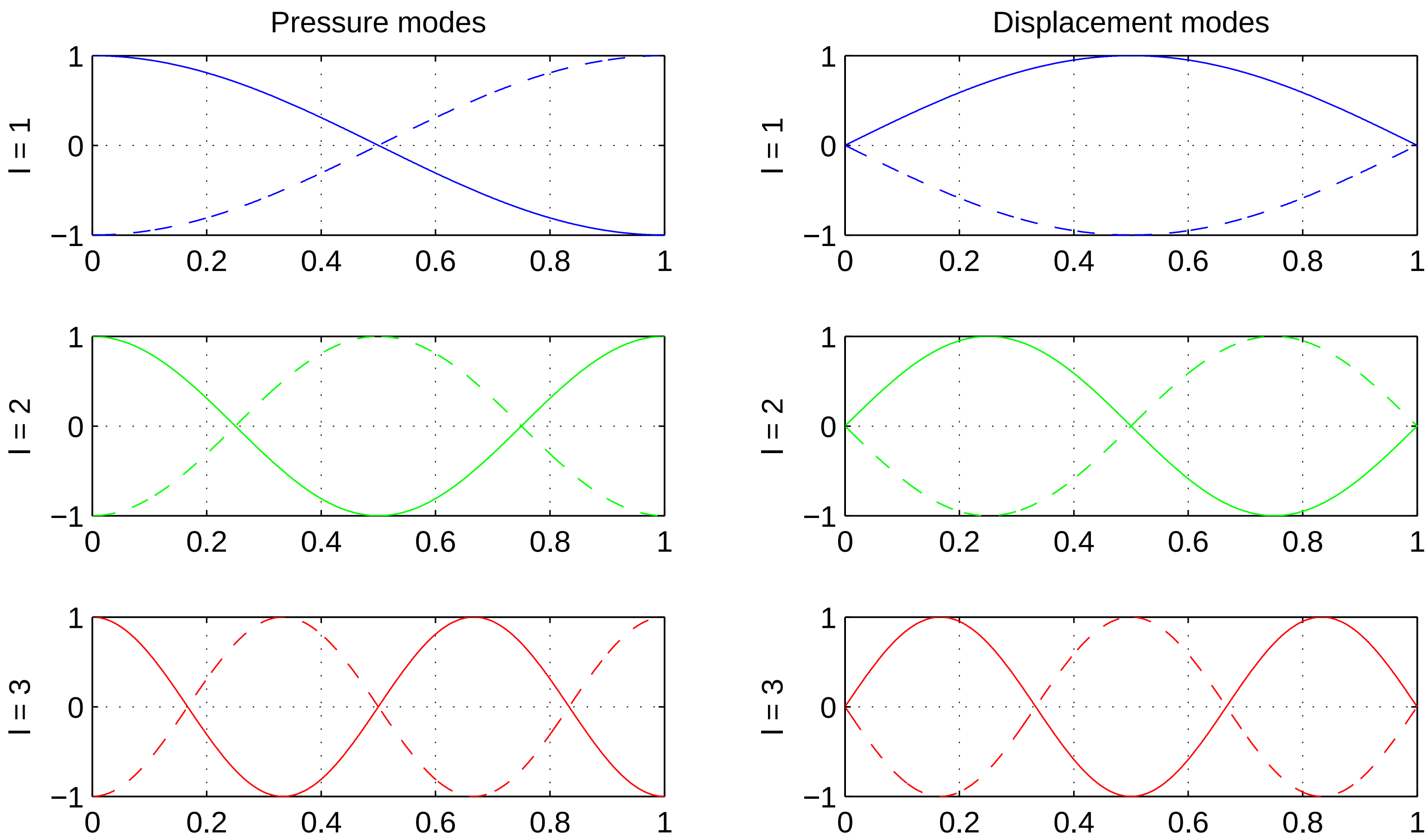
which can be expressed in various ways,

$$k = \frac{n\pi}{L} \quad \text{and} \quad \omega = \frac{n\pi c}{L} \quad \text{and} \quad f = \frac{nc}{2L}$$

At resonance, we obtain pressure mode shapes of the form

$$\begin{aligned} \mathbf{p}(t, x) &= 2\mathbf{p}^{(+)}\, X(x)\, \mathbf{e}^{j\omega t} \\ \text{where} \qquad X(x) &= \cos\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{10}$$

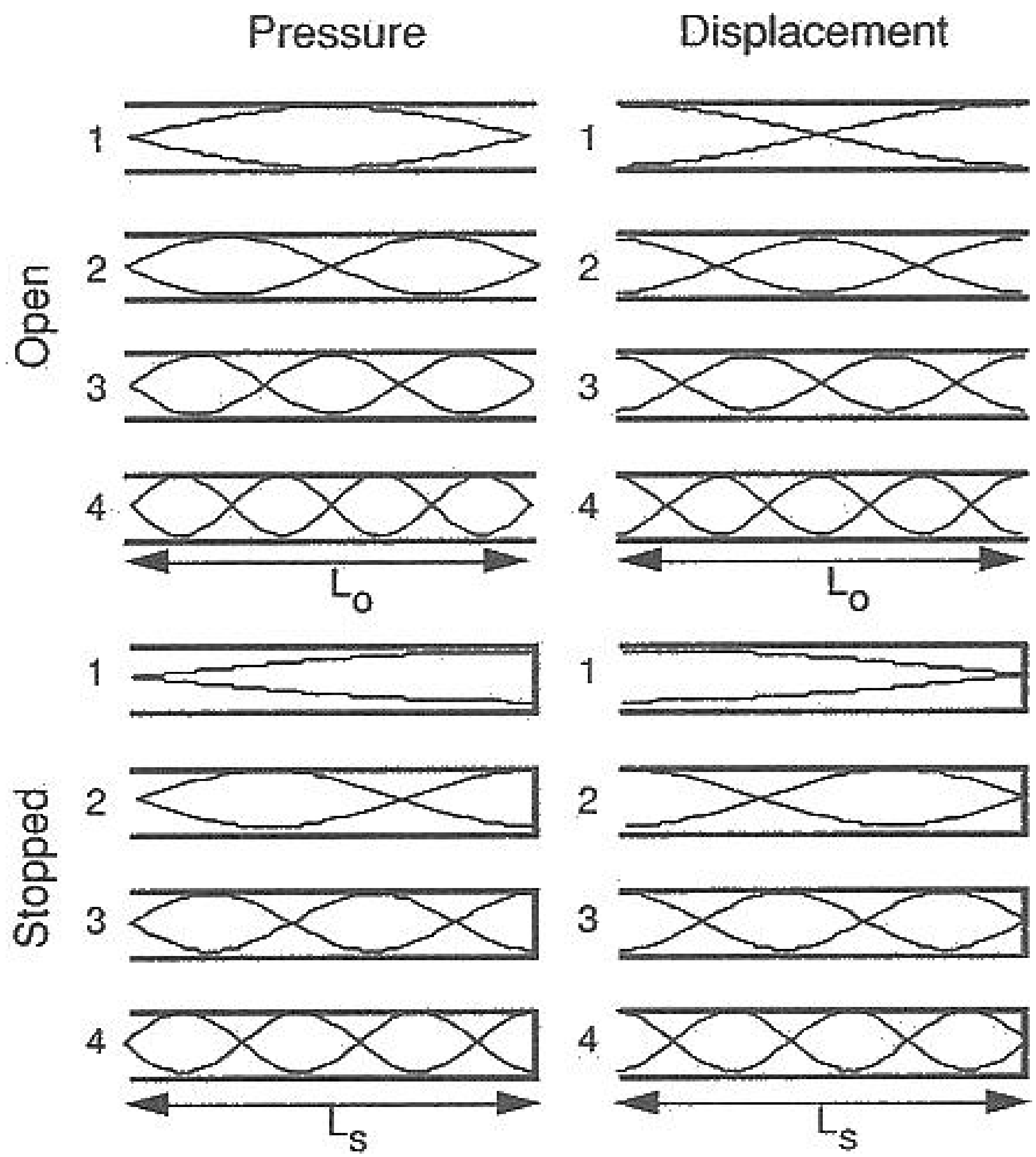
Pressure and displacement/velocity modes



Considering number of wavelengths that fit the return path, we get:

$$\boxed{f(l) = \frac{l\,c}{2L_C}} \tag{11}$$

Modes in open and semi-closed pipes



$$f(n) = \frac{n\,c}{2L_O}$$

$$f(m) = \frac{(2m-1)c}{4L_S}$$

Rectangular waveguide

Given four rigid walls in x and y directions, and open ends in the z direction that allow acoustic energy to propagate down the waveguide, we obtain solutions of the form:

$$p(x, y, z, t) = A \cos k_x x \, \cos k_y y \; \mathrm{e}^{j(\omega t - k_z z)} \tag{12}$$

with $k_x = \frac{l\pi}{L}$, $k_y = \frac{m\pi}{W}$, and $k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(k_x^2 + k_y^2\right)}$

The longitudinal wavenumber k_z depends on the transverse wavenumbers, but is only real above the **cut-on frequency**, $f \geq f_{\text{cut-on}}$,

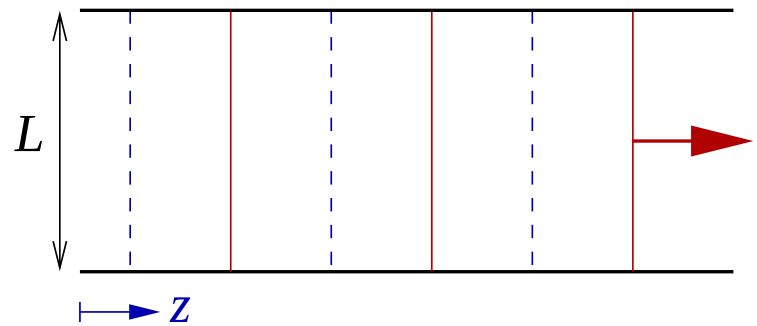
$$\boxed{f_{\text{cut-on}} = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2}} \tag{13}$$

otherwise with imaginary roots, tranverse mode becomes **evanescent** and does not propagate, i.e, waveguide only supports axial waves along z.

Transverse modes in a rectangular waveguide

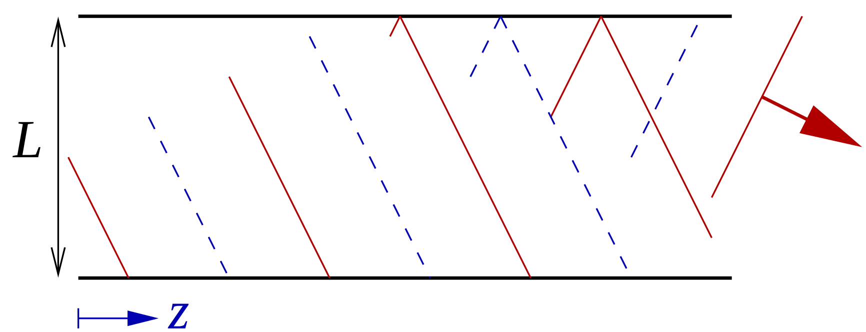
Plane wave propagation

zero transverse mode, for all f



Transverse wave propagation

non-zero cross mode, for $f \geq f_{\text{cut-on}}$



Summary of acoustic devices

- **cavities**
  - rectangular
- **resonators**
  - open, closed & semi-closed pipes
- **waveguides**
  - rectangular

Reference

L. E. Kinsler, A. R. Frey, A. B. Coppens and J. V. Sanders, "Fundamentals of Acoustics", 4th ed., Wiley, 2000. Chapters 9 & 10, [shelf 534 KIN]