EE1.el3 (EEE1023): Electronics III

Mechanics lecture 10
Oscillation

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Oscillation

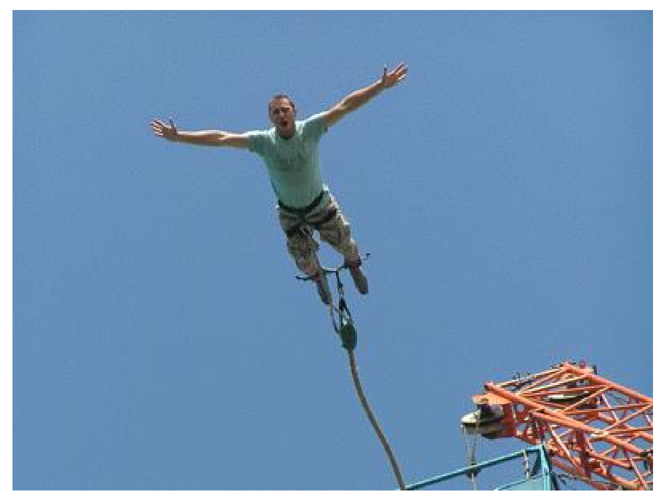
- Periodic oscillation
- Simple harmonic motion (SHM)

- Pendulum example
- Angular momentum
- Damped oscillations

Preparation for Oscillation

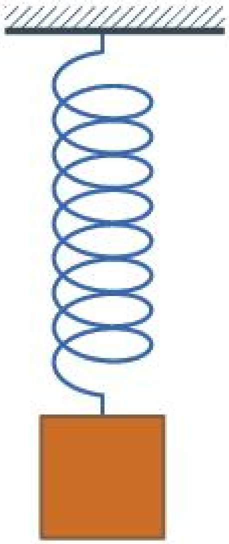
- What is **simple harmonic motion** (SHM)?
  - look up the equation of motion for SHM

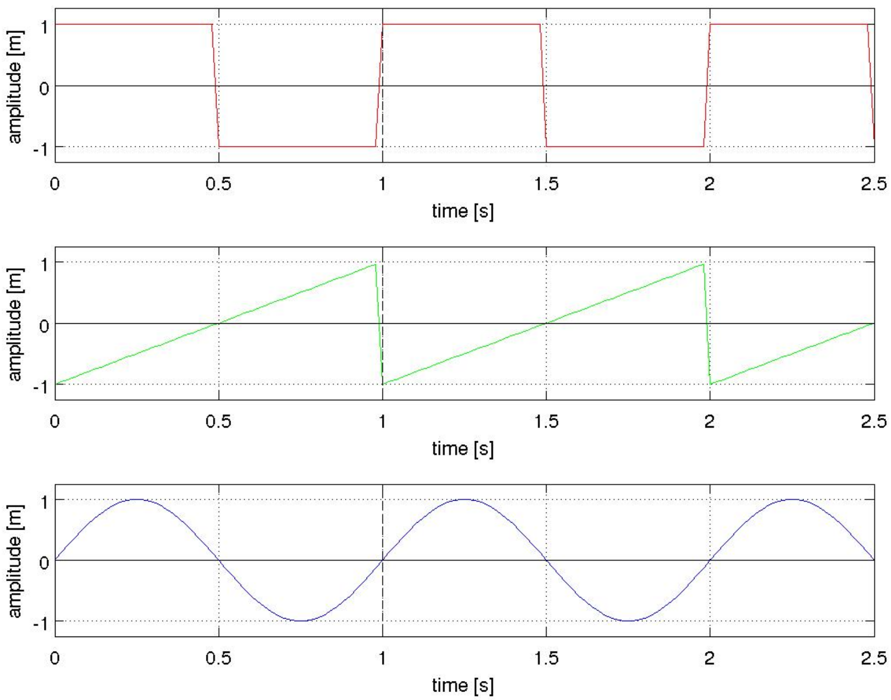
- Example of SHM
  - make a sketch of SHM
  - identify what forces are involved in your example



Periodic oscillation

- Period, T
- Frequency, $f = 1/T$
- Angular frequency, $\omega = 2\pi f$



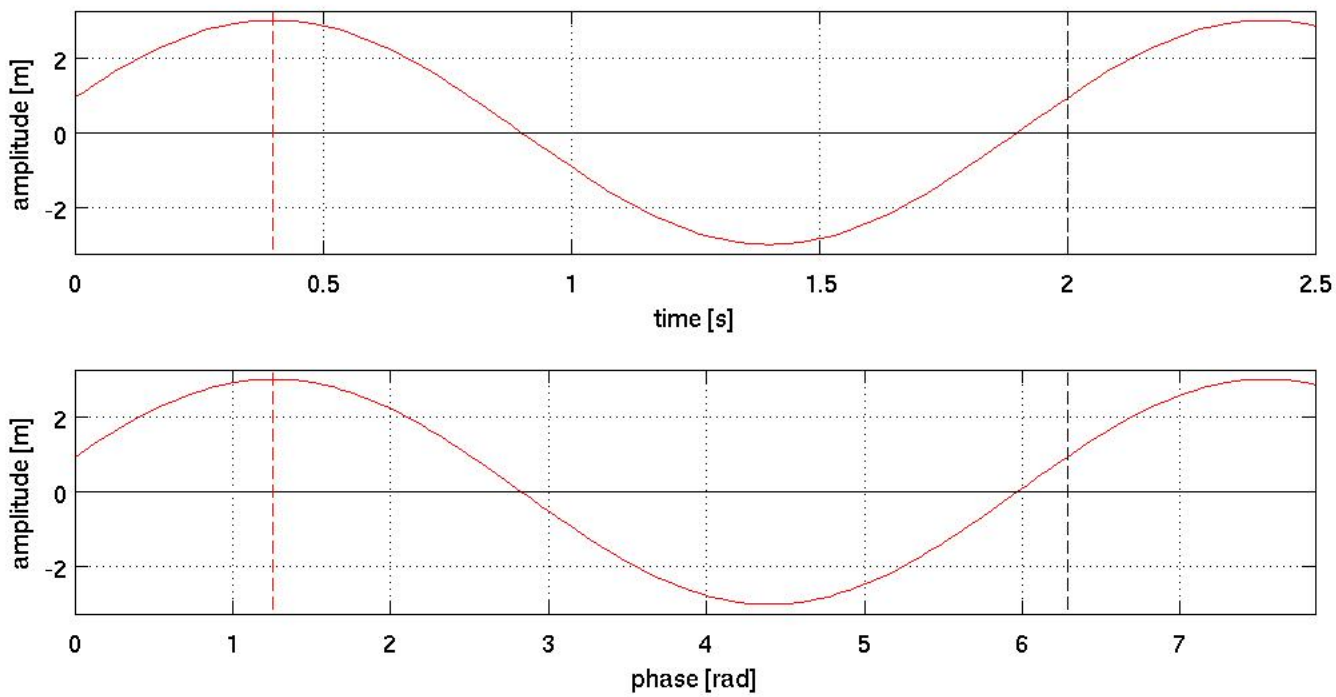


Kinematics of SHM

Consider a sinusoid, a pure harmonic oscillation about an equilibium position:

$$x(t) = A\cos(2\pi f t + \phi) \quad (1)$$

where $A = 3$, $f = 0.5\,\text{Hz}$ and $\phi = -72^\circ$



Dynamics of a mass on a spring

For this undamped 2nd-order system, the mass has inertia (Newton), and the spring provides an elastic force (Hooke):

$$\begin{aligned} m\ddot{x} &= -kx \\ \ddot{x} + \frac{k}{m}x &= 0 \end{aligned} \tag{2}$$

This gives us the SHM **equation of motion**:

$$\boxed{\ddot{x} + {\omega_n}^2 x = 0} \tag{3}$$

where the natural (angular) frequency, $\omega_n = \sqrt{k/m}$.

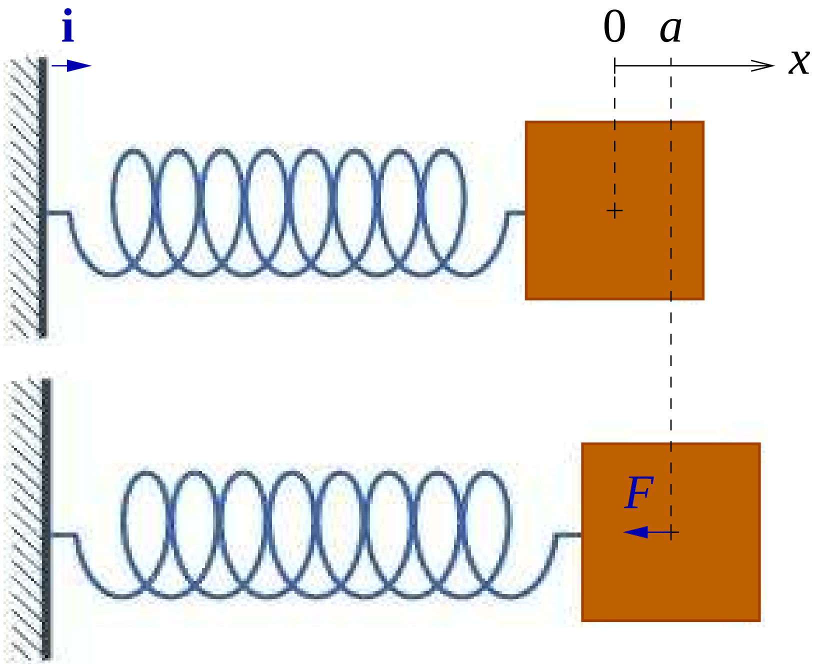
There are solutions to this equation of the form:

$$x(t) = X_0 \cos(\omega_n t + \phi_0) \tag{4}$$

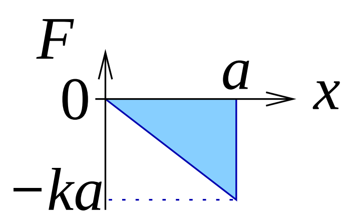
since $\ddot{x} = -\omega_n^2 x$.

Energy of an oscillating spring

Potential energy is the opposite of the work done by the spring force:



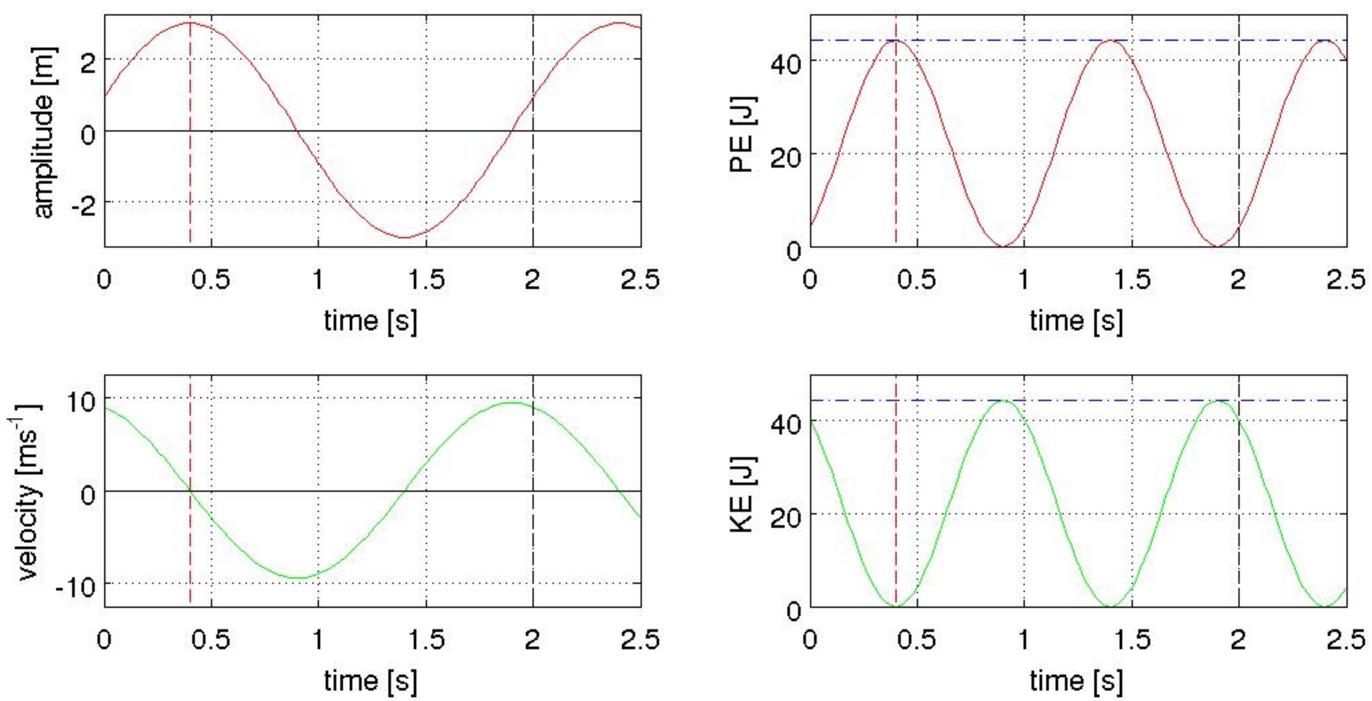
$$
\begin{aligned}
U &= -\int_0^a \mathbf{F}.\mathbf{dx} \\
&= \int_0^a (kx\mathbf{i}).(dx\mathbf{i}) \\
&= \int_0^a (kx)\, dx \\
&= \left[\frac{kx^2}{2}\right]_0^a \\
&= \frac{ka^2}{2} \qquad (5)
\end{aligned}
$$



Energy conservation

Undamped oscillation occurs when only conservative forces apply, so energy is conserved:

$$\begin{aligned} E &= W_K(t) + U(t) \\ &= \frac{1}{2} m\,\dot{x}(t)^2 + \frac{1}{2} k\,x(t)^2 \end{aligned} \tag{6}$$



Examples

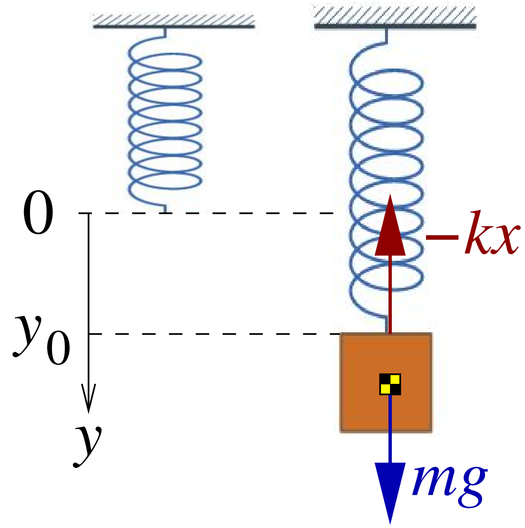
Vertical oscillations

At equilibrium position,

$$F_G + F_S = 0 \qquad (7)$$

$$mg - ky_0 = 0$$

$$\Rightarrow \qquad y_0 = \frac{mg}{k} \qquad (8)$$



Equation of motion:

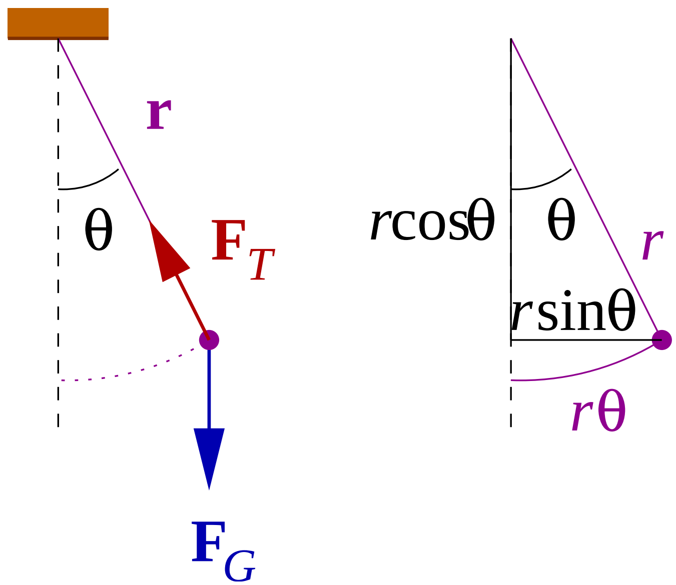
$$m\ddot{y} = mg - ky \qquad (9)$$

Substituting $y = y_0 + y'$ gives the familiar expression,

$$m\ddot{y} = mg - k\left(y_0 + y'\right) = mg - mg - ky'$$

$$\Rightarrow \qquad \boxed{m\ddot{y}' + ky' = 0} \qquad (10)$$

Pendulum by force



For small angles:

$$\begin{aligned} \sin\theta &\approx \theta \\ \cos\theta &\approx 1 \end{aligned} \qquad (11)$$

Balancing tangential forces and approximating, gives

$$-mg\sin\theta = m\left(r\ddot{\theta}\right) \qquad (12)$$

$$\ddot{\theta} + \frac{g}{r}\theta \approx 0 \qquad (13)$$

which has natural frequency $\omega_n = 2\pi/T$, and period

$$T = 2\pi\sqrt{\frac{r}{g}} \qquad (14)$$

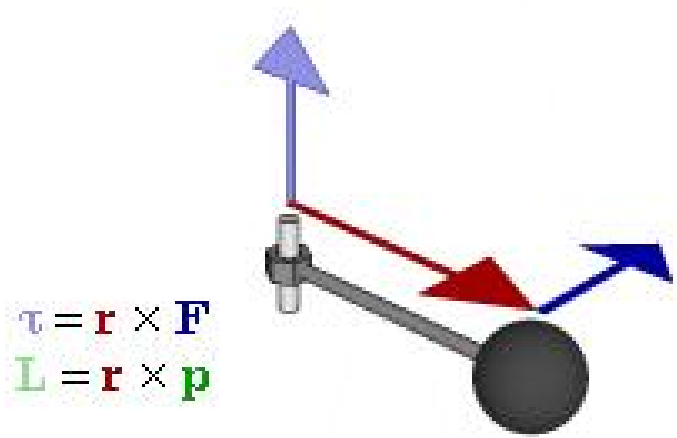
Angular momentum

We recall from last time:

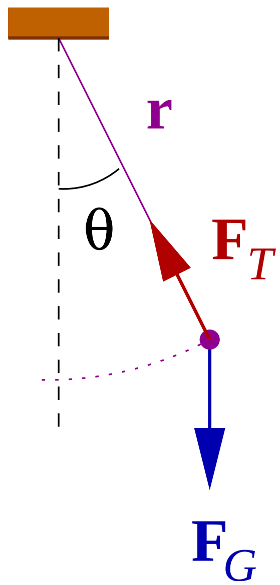
- Torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_G$
- Linear momentum, $\mathbf{p} = m\dot{\mathbf{r}}$
- Angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

From Newton's 2nd law, we consider changes in moment of momentum:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \qquad (15)$$



Pendulum by torque

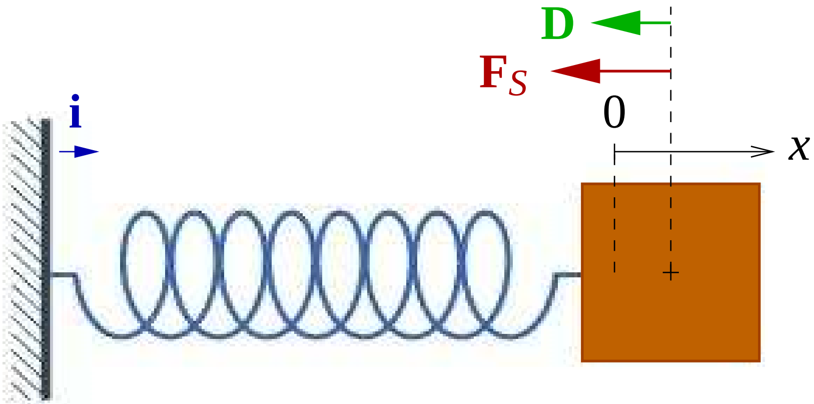


$$\begin{aligned}\boldsymbol{\tau} &= \frac{d\mathbf{L}}{dt} \\ &= \frac{d(\mathbf{r}\times\mathbf{p})}{dt} \\ &= \frac{d\mathbf{r}}{dt}\times(m\dot{\mathbf{r}}) + \mathbf{r}\times\frac{d(m\dot{\mathbf{r}})}{dt} \\ &= m\,\mathbf{r}\times\ddot{\mathbf{r}} \end{aligned} \tag{16}$$

Balancing gravitational restoring torque about the pivot with the inertial torque gives the expected result, as in (12):

$$\begin{aligned} -r\sin\theta\, mg &= mr^2\ddot{\theta} \\ \Rightarrow \quad -mg\sin\theta &= mr\ddot{\theta} \end{aligned} \tag{17}$$

Oscillation of a mass on a spring with damping



Spring force:

$$\mathbf{F}_S = -k\mathbf{x}$$

Damping force:

$$\mathbf{D} = -b\dot{\mathbf{x}}$$

Equating the inertial force to the sum of these forces yields the equation of motion for damped oscilation:

$$m\ddot{\mathbf{x}} = -b\dot{\mathbf{x}} - k\mathbf{x}$$

$$\Rightarrow \qquad \ddot{\mathbf{x}} + \frac{b}{m}\dot{\mathbf{x}} + \frac{k}{m}\mathbf{x} = 0 \tag{18}$$

which has solutions of the form

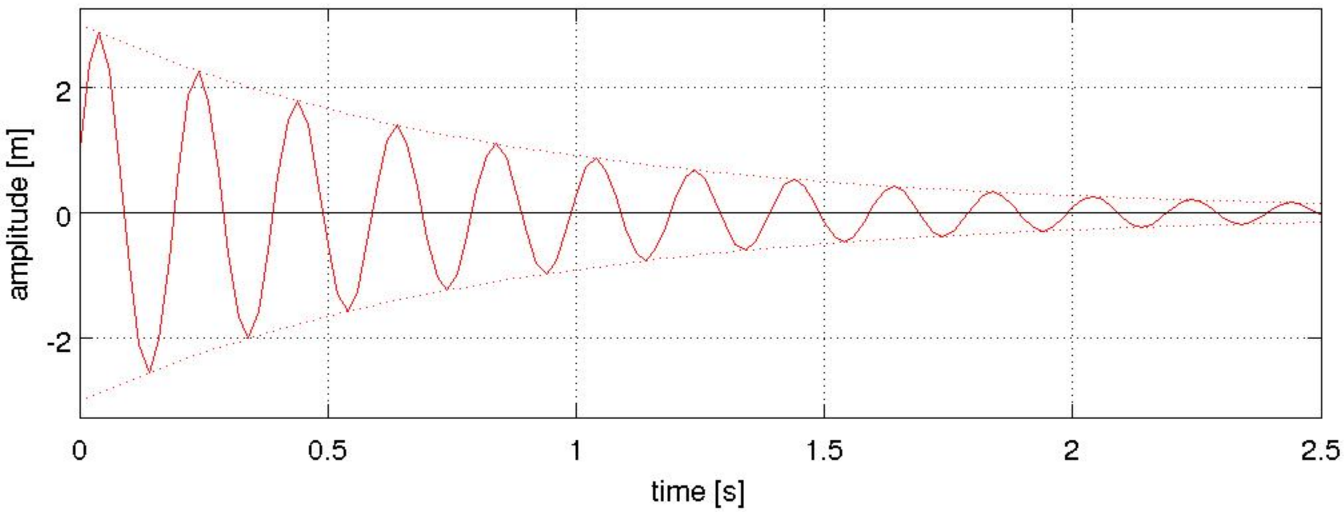
$$\mathbf{x}(t) = \mathbf{X}_0\,\mathbf{e}^{-bt/2m}\cos\left(\omega_d t + \phi_0\right) \tag{19}$$

where the damped frequency is $\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Damped oscillation

Oscillation **envelope** decays over time as exp $-bt/2m$, at a slightly lower rate of vibration than the natural frequency, $\omega_d \leq \omega_n$:

$$\mathbf{x}(t) = \mathbf{X}_0\,\mathbf{e}^{-bt/2m}\,\cos\left(\omega_d t + \phi_0\right)$$



Summary of SHM

- Periodic oscillation
  - period T, frequency f and angular frequency ω
- Simple harmonic motion (SHM)
  - conservation of energy E
  - natural frequency ω_n
- Pendulum example
  - small angle approximation, $\theta \ll \pi/2$
- Angular momentum
  - moment of momentum $\mathbf{L}$
- Damped oscillations
  - damped frequency ω_d