EE1.el3 (EEE1023): Electronics III

Mechanics lecture 9
Rigid body dynamics

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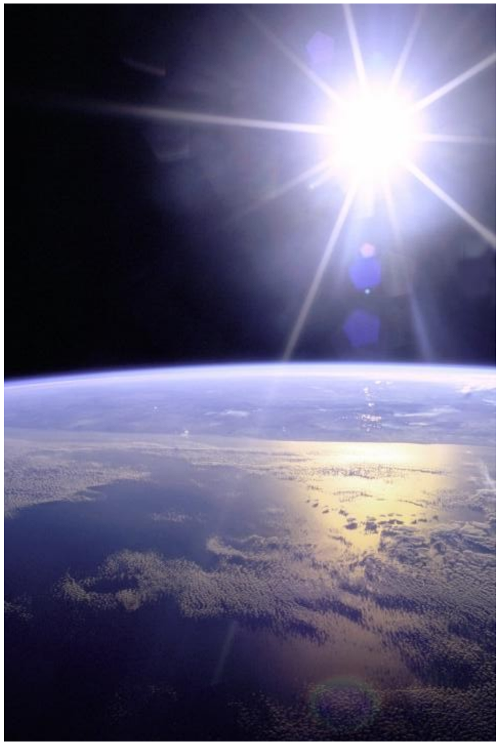
Rotational dynamics

- Dynamics of circular motion
- Angular momentum
- Angular inertia
- Angular work & power
- Kepler's law
- Orbiting the earth



Preparation for Rigid body dynamics

- What is **angular momentum**?
  - look up the equation for it
  - define all the terms in the equation

- What is **angular inertia**?
  - look up its definition
  - give one example

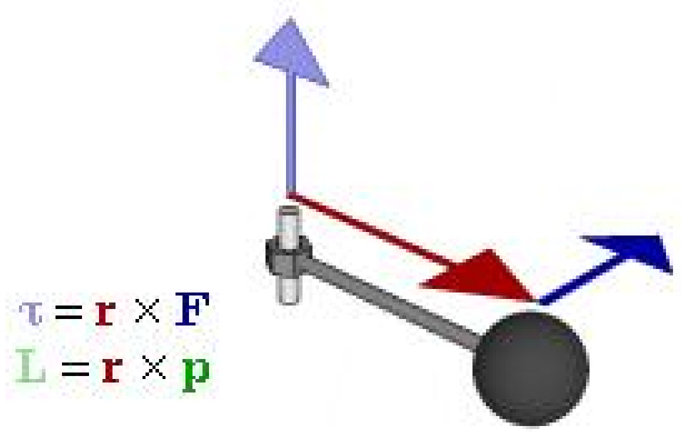
Angular momentum

Angular momentum represents the *moment of momentum*

- Torque, the moment of force, is $\mathbf{T} = \mathbf{r} \times \mathbf{F}$
- Angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
  where linear momentum $\mathbf{p} = m\,\dot{\mathbf{r}}$

From Newton's 2nd law, we consider changes in moment of momentum:

$$\mathbf{T} = \frac{d\mathbf{L}}{dt} \qquad (1)$$



Angular inertia

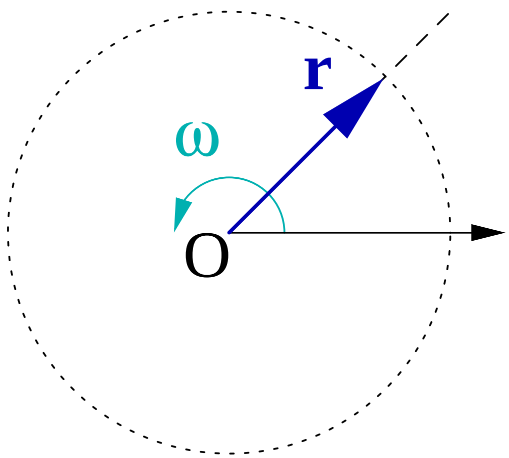
Angular inertia represents the *moment of inertia*

- For a mass in circular motion, the angular inertia is
$$J = r^2 m$$

- Torque creates angular acceleration
$$\mathbf{T} = J\ddot{\boldsymbol{\theta}} \qquad (2)$$
which is the rate of change of angular velocity, $\ddot{\boldsymbol{\theta}} = \dot{\boldsymbol{\omega}}$

This gives us neat expression for angular momentum:

$$\mathbf{L} = J\boldsymbol{\omega} \qquad (3)$$



Moments of inertia

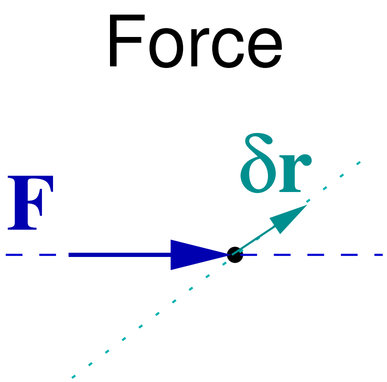
The moments of inertia of some common objects are listed with the axis through their centre of mass:

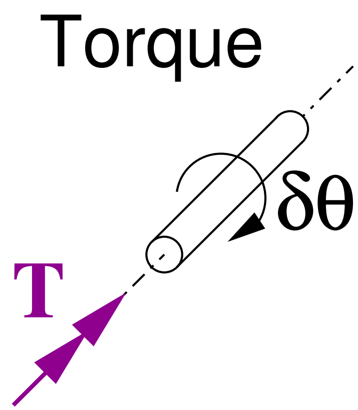
object	J
rod of length l	$\frac{ml^2}{12}$
solid disc of radius r	$\frac{mr^2}{2}$
hollow sphere of radius r	$\frac{2mr^2}{3}$
solid sphere of radius r	$\frac{2mr^2}{5}$

Work

The **work** done is equal to:

1. the force times the distance along its line of application
2. the parallel force component times the displacement





Linear work is $W_F = \mathbf{F}.\boldsymbol{\delta}\mathbf{r}$ Rotational work is $W_T = \mathbf{T}.\boldsymbol{\delta\theta}$

In general, the **total work** is their sum:

$$W = W_F + W_T \tag{4}$$

Power

Power is the rate of work

Average power over a short time interval δt is

$$P_F = \mathbf{F}.\frac{\boldsymbol{\delta}\mathbf{r}}{\delta t} \qquad\qquad P_T = \mathbf{T}.\frac{\boldsymbol{\delta\theta}}{\delta t}$$

Instantaneous power is the limit as $\delta t \to 0$:

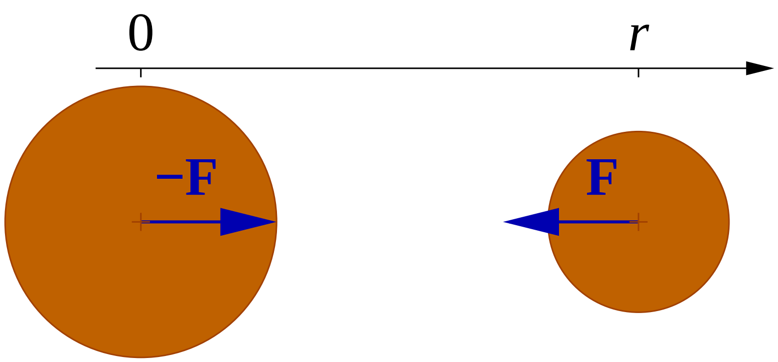
$$\begin{aligned} P_F &= \mathbf{F}.\frac{d\mathbf{r}}{dt} \\ &= \mathbf{F}.\dot{\mathbf{r}} \end{aligned} \qquad (5) \qquad\qquad \begin{aligned} P_T &= \mathbf{T}.\frac{d\boldsymbol{\theta}}{dt} \\ &= \mathbf{T}.\boldsymbol{\omega} \end{aligned} \qquad (6)$$

In general, the **total power** is their sum:

$$P = P_F + P_T \qquad (7)$$

Examples

Motion in a conservative force field



Gravitational force:

$$F = \frac{GMm}{r^2} \qquad (8)$$

where

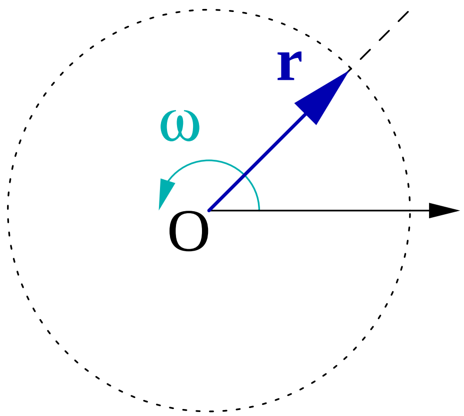
$G = 6.67 \times 10^{-11}\,\mathrm{N\,kg^{-2}}$

Including the direction of the force, we get the vector force on the smaller mass:

$$\mathbf{F}_G = \frac{GMm}{r^2}(-\mathbf{e}_r) \qquad (9)$$

where the radial unit vector is defined $\mathbf{e}_r = \frac{\mathbf{r}}{r}$.

Inertial force for circular motion



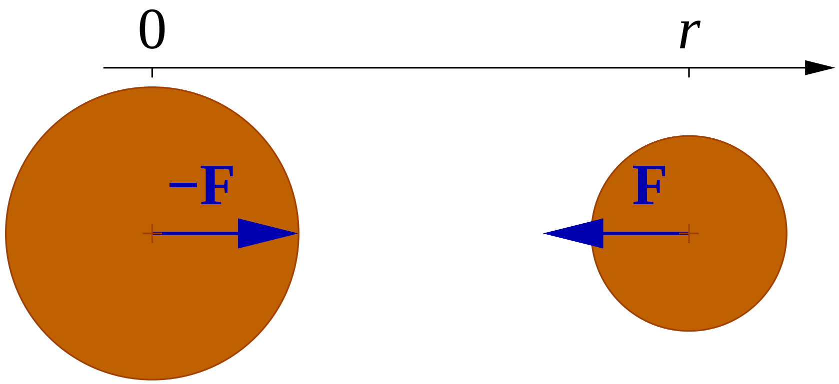
The kinematic expression for circular motion gives us

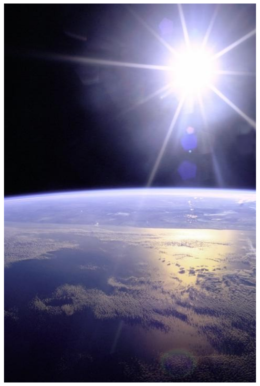
$$\begin{aligned}
\mathbf{r}(t) &= r\cos\omega t\,\mathbf{i} + r\sin\omega t\,\mathbf{j} \\
\dot{\mathbf{r}}(t) &= -\omega\, r\sin\omega t\,\mathbf{i} + \omega\, r\cos\omega t\,\mathbf{j} \\
\ddot{\mathbf{r}}(t) &= -\omega^2 r\cos\omega t\,\mathbf{i} - \omega^2 r\sin\omega t\,\mathbf{j} \\
&= -\omega^2\,(r\cos\omega t\,\mathbf{i} + r\sin\omega t\,\mathbf{j}) \\
&= -\omega^2\mathbf{r}(t) \qquad (10)
\end{aligned}$$

So the inertial force needed to maintain circular motion is

$$\mathbf{F}_K = m\ddot{\mathbf{r}} = -m\omega^2\mathbf{r} \qquad (11)$$

Kepler's 3rd law for planetary orbits





Considering radial forces of a planet orbiting the sun:

$$F_G = \frac{GM_\odot}{r^2} m = F_K = m\omega^2 r = m\left(\frac{2\pi}{T}\right)^2 r \qquad (12)$$

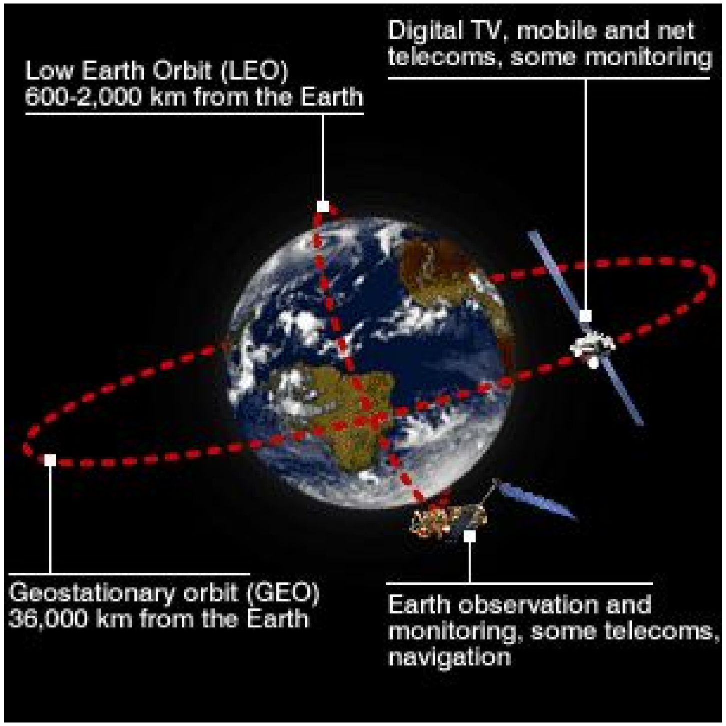
The period T depends on the radial distance r as

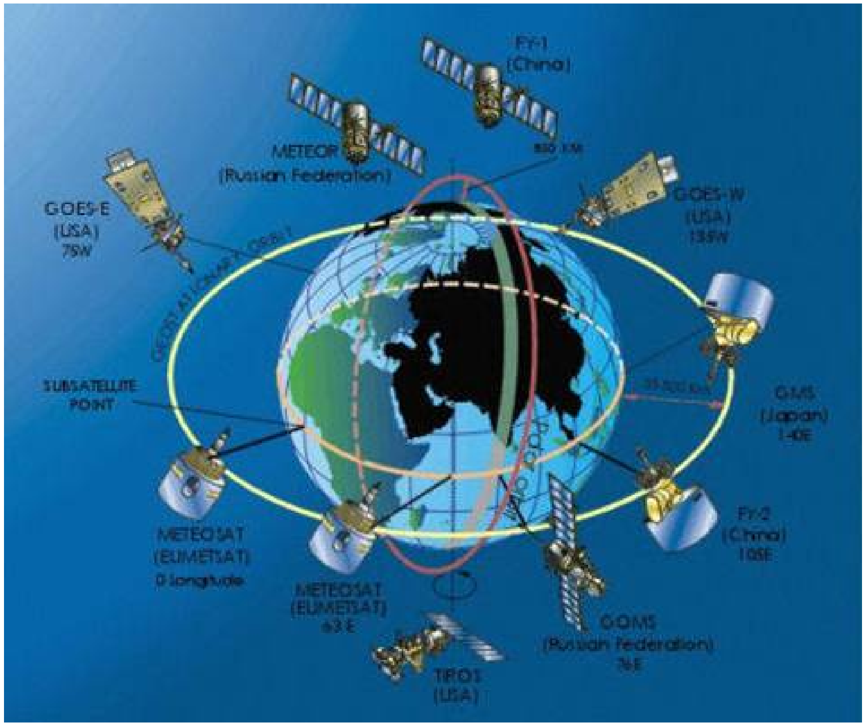
$$T^2 = \frac{4\pi^2}{GM_\odot} r^3 \qquad (13)$$

$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ and $M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$

Low earth orbit (LEO)

For satellites orbiting close to the earth's surface





$$T^2 = \frac{4\pi^2}{GM_\oplus} r^3 \qquad (14)$$

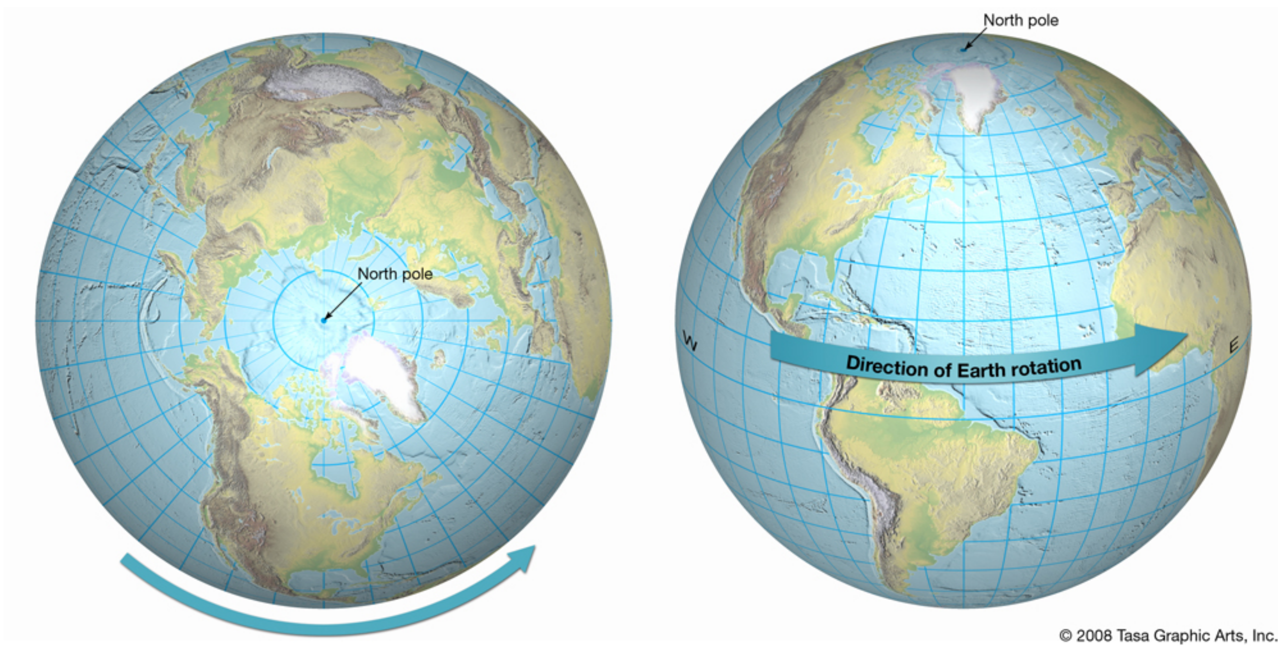
$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ and $M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}$

Geostationary earth orbit (GEO)

To match the earth's spin, the period is set to 24 hours

$$T_{\mathrm{GEO}}{}^2 = \frac{4\pi^2}{GM_\oplus}\, r_{\mathrm{GEO}}{}^3 \tag{15}$$

where $T_{\mathrm{GEO}} = 24 \times 60 \times 60\,\mathrm{s}$, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}$



Using potential energy to calculate escape velocity

Potential energy to move from earth's surface into space is

$$U_G = -\int_{R_\oplus}^{\infty} \mathbf{F}.d\mathbf{r} = \int_{R_\oplus}^{\infty} \frac{GM_\oplus m}{r^2} dr$$
$$= \left[-\frac{GM_\oplus m}{r}\right]_{R_\oplus}^{\infty} = \frac{GM_\oplus m}{R_\oplus} \qquad (16)$$

Escape velocity is obtained by providing an equal amount of kinetic energy:

$$W_K = \frac{1}{2} m \, \dot{r}_{\mathrm{esc}}{}^2 = U_G$$
$$\Rightarrow \qquad \dot{r}_{\mathrm{esc}} = \sqrt{\frac{2GM_\oplus}{R_\oplus}} \qquad (17)$$

$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, $M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}$ and $R_\oplus = 6371\,\mathrm{km}$

Summary of Rotational dynamics

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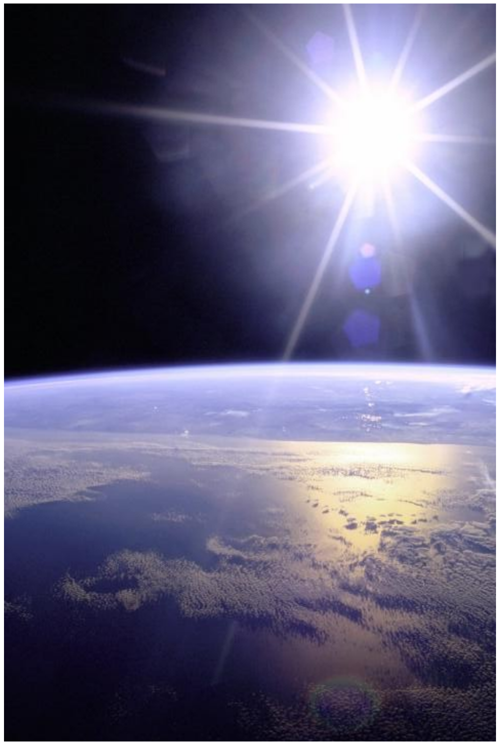

Preparation for Oscillation

• What is **simple harmonic motion** (SHM)?

  – look up the equation of motion for SHM

• Example of SHM

  – make a sketch of SHM

  – identify what forces are involved in your example