Mechanics lecture 7
Moment of a force, torque, equilibrium of a body

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http://www.ee.surrey.ac.uk/Teaching/Courses/ee1.el3/
Moments, torque and general equilibrium

• Moments of a force
  – vector product equation
  – moment about a point

• Torque as a vector
  – properties of a couple
  – components of a torque

• Equilibrium of a body
  – forces in balance
  – torques in balance

• Preparation
  – What is the moment of a force? *Give the equation for a moment*
  – What is an example of torque? *Find a definition of torque*
  – What does direction of torque mean? *Illustrate with an example*
G.3

Moment of a force about a point

• A moment is a rotating action of a force about a pivot point
• It tends to turn a body about an axis:
  – the axis is \( \perp \) to the force and its displacement from the pivot
  – the axis does not intersect with the force’s line of action and is not \( \| \) to the force

• Moment of a force \( \mathbf{P} \) about point \( O \) is defined:

\[
\mathbf{M}_O = d \mathbf{P} \mathbf{e} \quad \text{(Nm)}
\]

where \( d = r \sin \alpha \)

• It measures of the turning effect of \( \mathbf{P} \)
• Using the right-hand rule, the direction of \( \mathbf{M}_O \) is as \( \mathbf{e} \)
Moment expressed as a vector

- The line of action of force is defined by its point of application and direction.
- Force $\mathbf{F}$ applied at position $\mathbf{r}$ yields a moment about $O$:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where $\times$ denotes the vector product of $\mathbf{r}$ and $\mathbf{F}$ ($\neq \mathbf{F} \times \mathbf{r}$, opposite!)

- The cross product is simply the product when displacement $\mathbf{r}$ is perpendicular to force $\mathbf{F}$: i.e., $\mathbf{M} = \mathbf{r} \mathbf{F}_\perp$

otherwise, it depends on the angle $\theta$:

$$\mathbf{M} = \mathbf{r} \mathbf{F} \sin \theta$$
Ways of calculating moments

- As seen from the moment equation, there is an equivalence

\[ M = (r \sin \theta) F = r (F \sin \theta) \]

which allows two interpretations for the size of the moment:

1. it is the perpendicular distance from the pivot point to the line of action multiplied by the magnitude of the force
2. it is the displacement multiplied by component of the force perpendicular to it
Moments about three orthogonal axes

• The three directions of rotation are commonly called pitch, roll and yaw.

• Expressing torque in vector format with a cartesian reference frame, we get:

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} = \begin{bmatrix}
& \text{pitch} \\
& \text{roll} \\
& \text{yaw}
\end{bmatrix}
\]

• These components can be calculated directly in the vector product, e.g.

\[
M = r \times F = \begin{bmatrix}
3 \\
-1 \\
-2
\end{bmatrix} \times \begin{bmatrix}
0 \\
70 \\
0
\end{bmatrix} = \begin{bmatrix}
140 \\
0 \\
210
\end{bmatrix}
\]

Note:

<table>
<thead>
<tr>
<th>i × j = k</th>
<th>j × k = i</th>
<th>k × i = j</th>
</tr>
</thead>
<tbody>
<tr>
<td>j × i = -k</td>
<td>k × j = -i</td>
<td>i × k = -j</td>
</tr>
</tbody>
</table>
Moment of a couple: torque

- A couple is a pair of parallel, equal and opposite forces.
- The lines of action are not co-linear, separated by a distance, $d$.
- Although the resultant force is zero, it is not in equilibrium.
- Moment’s magnitude is $dP$ and direction $\perp$ to the plane of forces (right-hand rule).

For constant $d \times P$, the moment becomes a torque as $d \to 0$:

$$T = d \times P$$
Components of a torque

• As a special kind of moment, torque $\mathbf{T}$ can be resolved into orthogonal components too

\[ \mathbf{T} = T_x \mathbf{i} + T_y \mathbf{j} + T_z \mathbf{k} \]

• Component of $\mathbf{T}$ about an arbitrary axis AB through the Origin, in direction $\mathbf{e}$, is

\[ T_{AB} = (\mathbf{T} \cdot \mathbf{e}) \mathbf{e} \]
Properties of torque: pivot point

- Equivalent force combinations can yield the same resultant:

\[ M = M_1 + M_2 + \ldots + M_n \]

- We can use this property to shift the pivot point and simplify

- The resultant moment is sum of all moments about one point:

\[ M = \sum_{i=1}^{n} r_i \times P_i \]
Resultant of a force and torque system

- The **resultant** is the sum of all forces and moments on a body:

\[ R = P_1 + P_2 + P_3 + \ldots + P_n \]

\[ R = \sum_{i=1}^{n} P_i \]

\[ M = M_1 + M_2 + \ldots + M_n \]

\[ M = \sum_{i=1}^{n} r_i \times P_i \]

- We can substitute them without changing external effect on it
Equilibrium of a rigid body

- In general, equilibrium occurs when all forces and torques acting on a body are in balance.
- The resultant force and moment are both zero.
- There is no acceleration of the body.

\[ R = \sum_{i=1}^{n} P_i = 0 \]
\[ M = \sum_{i=1}^{n} r_i \times P_i = 0 \]
Free body diagrams

- A free body diagram shows a body isolated from other bodies
- All external forces and torques acting on it are shown

To solve for a body in equilibrium:
1. Decide on the body of interest
2. Draw a diagram of the body isolated from other bodies in contact with it
3. Show all forces and torques acting on the body (Newton’s 3rd law of opposition)
4. Select an appropriate reference frame
5. Establish all vectors and unit vectors
6. Apply equilibrium conditions
Equilibrium of a rigid body example

- Equilibrium occurs when the resultant force and the resultant moment are both equal to zero:

$$\sum_{(i)} \vec{F}_{(i)} = \vec{0} \quad \sum_{(i)} \vec{M}_{(i)} = \vec{0}$$
Example problem: gearbox in equilibrium

You now have all the tools to solve a problem like this!
Gearbox example: torque component about O

**Problem:**
If the gearbox is a 1-m cube with CoG at its centre, write down an expression for the resultant torque component $T_X$ about O, where $T_X = T.i$

**Solution:**

$$T_X = \frac{1}{2} W - R_C + T_{in}$$
Resultant moment example

- A yacht sailing windward (B) experiences ‘weather helm’ which makes the helmsman steer to prevent turning into the wind.
- Calculate the torque required from the rudder to hold its course.

\[ T_L = r \times L \]

\[
\begin{pmatrix}
1/2 \\
2 \\
0
\end{pmatrix} \times 
\begin{pmatrix}
0 \\
0 \\
4200
\end{pmatrix} = 
\begin{pmatrix}
8400 \\
-2100 \\
0
\end{pmatrix}
\]
Kinematics (1): rigid body statics

• **Moments**
  – moment of a force about a point
  – calculation using the vector product
  – torque induced by a couple
  – components of torque about an axis

• **Resultant moment**
  – sum of all moments about one point
  – can be used to simplify problems

• **Equilibrium**
  – free body diagram
  – resultant force and resultant moment equal to zero
Kinematics (2)

• **Motion of a rigid body**

• How do you define angular velocity?
  – How can we calculate it?
  – How can we express it as a vector?

• What are the relationships between angular displacement, angular velocity and angular acceleration?
  – write mathematical expressions of the relationships in both directions